

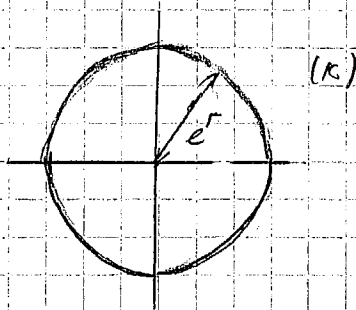
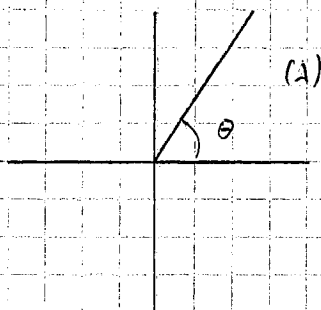
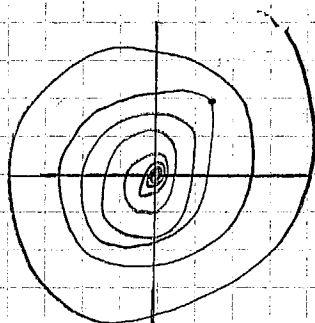
תרגיל 4 וסיקור מרחב המרוכב

(1) פונקציה המיוקרת  $f(z) = e^z$  כאשר:

$$e^z = e^r \cdot e^{iy} \quad \text{שם } y = \text{Im}(z) \in \mathbb{R} \quad , \quad \text{Re}(z) = r \quad (1)$$

$$e^z = e^x \cdot e^{i\theta} \quad \text{שם } x = \text{Re}(z) \in \mathbb{R} \quad , \quad \text{Im}(z) = \theta \quad (2)$$

$$e^z = e^x \cdot e^{iKx} \cdot e^{ib} \quad \Leftrightarrow \quad y = Kx + b \quad , \quad z = x + iy \quad (3)$$



(2) פונקציה המיוקרת  $f(z) = \frac{1}{z}$  (2)

$$z \in S_a \Leftrightarrow |z|^2 = a \cdot \text{Re}(z) \quad \Leftrightarrow \quad x^2 + y^2 = ax \quad (1)$$

$$\Leftrightarrow x^2 - ax + \frac{1}{4}a^2 + y^2 = \frac{1}{4}a^2 \Leftrightarrow (x - \frac{1}{2}a)^2 + y^2 = (\frac{1}{2}a)^2$$

כדור, מרחק מהנקודה  $(\frac{1}{2}a, 0)$  הוא  $\frac{1}{2}a$  ורדיוס  $\frac{1}{2}a$

$$f(z) = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} = \frac{x}{ax} - i \frac{y}{ax} = \frac{1}{a} - i \frac{y}{x}$$

אם  $a \neq 0$  והנקודה  $(\frac{1}{2}a, 0)$  היא הנקודה

אם  $a = 0$  - נקודה  $(0, 0)$  - נקודה הנקודה  $\frac{1}{a}$

$$z \in L_b \Leftrightarrow \text{Im}(z) = \text{Re}(z) + b \Leftrightarrow z = x + i(x + b) \quad (2)$$

זהו ישר עם שיפוע 1 הנעבר דרך הנקודה  $i b$

$$w = \alpha + i\beta \quad \text{שם } z = \frac{1}{w} \quad \text{שם } w = f(z) = \frac{1}{z} \quad (3)$$

$$z \in L_b \Leftrightarrow \frac{1}{w} = \frac{\alpha}{\alpha^2 + \beta^2} - i \frac{\beta}{\alpha^2 + \beta^2} \in L_b \Leftrightarrow -\frac{\beta}{\alpha^2 + \beta^2} = \frac{\alpha}{\alpha^2 + \beta^2} + b$$

$$\Leftrightarrow \frac{\alpha + \beta}{\alpha^2 + \beta^2} + b = 0 \Leftrightarrow b(\alpha^2 + \beta^2) + \alpha + \beta = 0$$

$$0 = \alpha^2 + \frac{1}{b}\alpha + \beta^2 + \frac{1}{b}\beta = (\alpha + \frac{1}{2b})^2 + (\beta + \frac{1}{2b})^2 - \frac{1}{2b^2} \quad : b \neq 0$$

$$\Leftrightarrow (\alpha + \frac{1}{2b})^2 + (\beta + \frac{1}{2b})^2 = (\frac{1}{\sqrt{2}b})^2 \quad (b \in \mathbb{R})$$

שם  $w$  נמצא על מעגל בעל רדיוס  $\frac{1}{\sqrt{2}|b|}$  ונקודה  $(-\frac{1}{2b}, -\frac{1}{2b})$

$$\text{שם } w \text{ נמצא על ישר } \alpha = -\beta \Leftrightarrow \frac{\alpha + \beta}{\alpha^2 + \beta^2} = 0 \quad : b = 0$$

אם שיפוע  $-1$  הנקודה  $(0, 0)$

$$\varphi_1, \varphi_2: [0,1] \rightarrow [0,1] \text{ "ק" } \quad \gamma_2 \sim \gamma_4, \gamma_1 \sim \gamma_3 \quad (1) (3)$$

$$\varphi_1(0) = \varphi_2(0) = 0 \quad \text{ע"פ } \gamma_1 \text{ ו-} \gamma_2 \text{ ממוסגרים}$$

$$\gamma_3 = \gamma_4 \circ \varphi_2, \gamma_1 = \gamma_2 \circ \varphi_1 \quad \text{ע"פ } \varphi_1(1) = \varphi_2(1) = 1 \quad / \Delta$$

$$\gamma_1 \gamma_3(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_3(2t-1) & \frac{1}{2} < t \leq 1 \end{cases} = \begin{cases} \gamma_2 \circ \varphi_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_4 \circ \varphi_2(2t-1) & \frac{1}{2} < t \leq 1 \end{cases} \quad \text{15/6}$$

$$\varphi(t) = \begin{cases} \frac{1}{2} \varphi_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \frac{1}{2} \varphi_2(2t-1) + \frac{1}{2} & \frac{1}{2} < t \leq 1 \end{cases} \quad \text{15/7, 15/8}$$

$$\varphi \in \text{ממוסגרים} \quad \text{ע"פ } \gamma_1 \text{ ו-} \gamma_2 \quad / \Delta$$

$$\begin{aligned} \gamma_2 \gamma_4 \circ \varphi(t) &= \gamma_1 \gamma_3(\varphi(t)) = \begin{cases} \gamma_2(2 \cdot \frac{1}{2} \varphi_1(2t)) & 0 \leq t \leq \frac{1}{2} \\ \gamma_4(2 \cdot \frac{1}{2} \varphi_2(2t-1) + 1 - 1) & \frac{1}{2} < t \leq 1 \end{cases} \\ &= \begin{cases} \gamma_2 \circ \varphi_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_4 \circ \varphi_2(2t-1) & \frac{1}{2} < t \leq 1 \end{cases} = \gamma_1 \gamma_3(t) \end{aligned}$$

$$\begin{aligned} \int_{\gamma_1 \gamma_3} f dz &= \int_0^1 f(\gamma_1 \gamma_3(t)) \cdot (\gamma_1 \gamma_3)'(t) dt \quad \gamma_1 \gamma_3 \sim \gamma_2 \gamma_4 \quad / \Delta \\ &= \int_0^{\frac{1}{2}} f(\gamma_1(2t)) \cdot 2 \cdot \gamma_1'(2t) dt + \int_{\frac{1}{2}}^1 f(\gamma_2(2t-1)) \cdot 2 \cdot \gamma_2'(2t-1) dt = \\ &= \left\{ \begin{array}{l} u=2t \quad v=2t-1 \\ du=2dt \quad dv=2dt \end{array} \right\} = \int_0^{\frac{1}{2}} f(\gamma_1(u)) \gamma_1'(u) du + \int_0^{\frac{1}{2}} f(\gamma_2(u)) \gamma_2'(u) du = \end{aligned}$$

$$= \int_{\gamma_1} f dz + \int_{\gamma_2} f dz$$

$$\int_{\gamma} (\lambda f + \mu g) dz = \int_0^{\frac{1}{2}} (\lambda f + \mu g)(\gamma(t)) \cdot \gamma'(t) dt = \quad (2)$$

$$\begin{aligned} &= \int_0^{\frac{1}{2}} (\lambda f(\gamma(t)) + \mu g(\gamma(t))) \cdot \gamma'(t) dt = \lambda \int_0^{\frac{1}{2}} f(\gamma(t)) \cdot \gamma'(t) dt + \\ &+ \mu \int_0^{\frac{1}{2}} g(\gamma(t)) \cdot \gamma'(t) dt = \lambda \int_{\gamma} f dz + \mu \int_{\gamma} g dz \end{aligned}$$

$$\text{מכאן } \int_{\gamma} (\lambda f + \mu g) dz = \lambda \int_{\gamma} f dz + \mu \int_{\gamma} g dz \quad \text{ע"פ } \gamma_1 \text{ ו-} \gamma_2 \text{ ממוסגרים}$$

$$\gamma[0, 2\pi] \rightarrow \mathbb{C} \quad \gamma(t) = 2e^{it} \quad \text{sonra } \gamma'(t) = 2ie^{it} \quad (4)$$

$$\int_{\gamma} \bar{z} dz = 2 \int_{\gamma} \operatorname{Re} z dt = 2 \int_0^{2\pi} 2 \cos t \cdot 2ie^{it} dt =$$

$$= 8i \int_0^{2\pi} e^{it} \cos t dt = 8i \int_0^{2\pi} e^{it} \frac{e^{it} + e^{-it}}{2} dt =$$

$$= 4i \int_0^{2\pi} (e^{2it} + 1) dt = 4i \cdot \left( \frac{1}{2i} e^{2it} + t \right) \Big|_0^{2\pi} =$$

$$= 2e^{4i\pi} - 2e^0 + 8i\pi - 0 = 8i\pi$$

$$\int_{\gamma} \bar{z}^2 - 2\bar{z} + 3 dz = \int_0^{2\pi} (4e^{2it} - 4e^{it} + 3) \cdot 2ie^{it} dt =$$

$$= \int_0^{2\pi} (8ie^{3it} - 8ie^{2it} + 6ie^{it}) dt = \frac{8}{3} e^{3it} - 4e^{2it} + 6e^{it} \Big|_0^{2\pi} =$$

$$= \frac{8}{3} - 4 + 6 - \frac{8}{3} + 4 - 6 = 0$$

$$\int_{\gamma} \bar{z}^4 dz = \int_0^{2\pi} 2^4 e^{-4it} \cdot 2ie^{it} dt = \frac{i}{8} \int_0^{2\pi} e^{-3it} dt =$$

$$= -\frac{1}{24} e^{-3it} \Big|_0^{2\pi} = -\frac{1}{24} + \frac{1}{24} = 0$$

$$\int_{\gamma} \operatorname{Re}(z) \cdot \operatorname{Im}(z) dz = \int_0^{2\pi} 2 \cos t \cdot 2 \sin t \cdot 2ie^{it} dt =$$

$$= 8i \cdot \int_0^{2\pi} \frac{e^{it} + e^{-it}}{2} \cdot \frac{e^{it} - e^{-it}}{2i} \cdot e^{it} dt =$$

$$= 2 \int_0^{2\pi} (e^{3it} + e^{it} - e^{it} - e^{-it}) dt = \frac{2}{3i} e^{3it} - \frac{2}{i} e^{-it} \Big|_0^{2\pi} =$$

$$= \frac{2}{3i} + \frac{2}{i} - \frac{2}{3i} - \frac{2}{i} = 0$$

$$\int_0^{2\pi} \cos^{2n} t dt = \int_0^{2\pi} 2^{-2n} (e^{it} + e^{-it})^{2n} dt =$$

(k) (5)

$$= 2^{-2n} \int_0^{2\pi} \sum_{k=0}^{2n} \binom{2n}{k} e^{i(2n-k)t} e^{-ikt} dt = 2^{-2n} \int_0^{2\pi} \sum_{k=0}^{2n} \binom{2n}{k} e^{i(2n-2k)t} dt =$$

$$= 2^{-2n} \sum_{k=0}^{2n} \binom{2n}{k} \int_0^{2\pi} e^{2i(n-k)t} dt \quad (\ominus)$$

$$\int_0^{2\pi} e^{imt} dt \stackrel{m \neq 0}{=} \frac{1}{im} e^{imt} \Big|_0^{2\pi} = 0$$

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$$\quad (\ominus) \quad 2^{-2n} \cdot \binom{2n}{n} \cdot \int_0^{2\pi} dt = \frac{2^{1-2n} \cdot \pi \cdot (2n)}{1}$$

$$\int_0^{2\pi} \frac{dt}{2 + \cos t} = \int_0^{2\pi} \frac{2dt}{4 + e^{it} + e^{-it}} = \int_0^{2\pi} \frac{2e^{it} dt}{e^{2it} + 4e^{it} + 1} =$$

(\Delta)

$$= \int_0^{2\pi} \frac{2e^{it} dt}{(e^{it} + 2)^2 - 3} = \left\{ \begin{array}{l} \gamma(t) = e^{it} + 2 \\ \gamma'(t) = ie^{it} \end{array} \right\} = \frac{2}{i} \int_{\gamma} \frac{dz}{z^2 - 3} = \frac{2}{i} \int_{\gamma} \frac{1}{2\sqrt{3}} \left( \frac{1}{z - \sqrt{3}} - \frac{1}{z + \sqrt{3}} \right) dz =$$

$$= \frac{1}{i\sqrt{3}} \int_{\gamma} \frac{dz}{z - \sqrt{3}} - \frac{1}{i\sqrt{3}} \int_{\gamma} \frac{dz}{z + \sqrt{3}} = \frac{1}{i\sqrt{3}} \cdot 2\pi i \cdot (n(\gamma, \sqrt{3}) - n(\gamma, -\sqrt{3})) = \frac{2\pi}{\sqrt{3}}$$

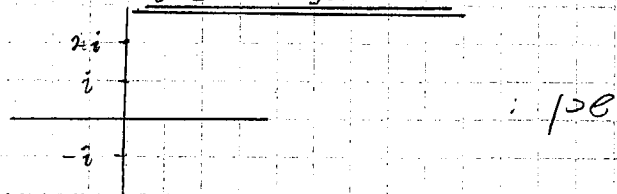
$$\gamma_r: [0, 2\pi] \rightarrow \mathbb{C}$$

$$\gamma_r(t) = 2i + re^{it}$$

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$$\int_{\gamma_r} \frac{dz}{z^2+1} = \int_{\gamma_r} \frac{1}{2i} \left( \frac{1}{z-i} - \frac{1}{z+i} \right) dz = \frac{1}{2i} \cdot 2\pi i \cdot (n(\gamma_r, i) - n(\gamma_r, -i))$$

$$= \pi \cdot \begin{cases} 0-0 & 0 \leq r < 1 \\ 1-0 & 1 < r < 3 \\ 1-1 & 3 < r \end{cases} = \begin{cases} \pi & 1 < r < 3 \\ 0 & \text{אחרת} \end{cases}$$



כל  $\gamma_r$  "x" נוקטת  $i, -i$  כל  $r < 1$  נ"ע

"x" נוקטת  $i$  רק  $1 < r < 3$  נ"ע. 0 נ"ע

הס"ס ה נ"ע  $-i$  כל  $r > 3$  נ"ע

כל  $r > 3$  נ"ע

$$\int_{\gamma_1} |z|^2 dz = \int_{\gamma_1} z \cdot \bar{z} dz = \int_0^{2\pi} (t + it^2)(t - it^2) \cdot (1 + 2it) dt = (k) (7)$$

$$= \int_0^{2\pi} (t^2 + t^4)(1 + 2it) dt = \int_0^{2\pi} t^2 + t^4 + 2it(t^3 + t^5) dt =$$

$$= \left[ \frac{1}{3} t^3 + \frac{1}{5} t^5 + 2i \left( \frac{1}{4} t^4 + \frac{1}{6} t^6 \right) \right]_0^{2\pi} = \frac{8}{15} + 2i \cdot \left( \frac{5}{12} \right) = \underline{\underline{\frac{8}{15} + \frac{5}{6} i}}$$

$$\gamma_2(t) = \cos 2t + i \sin 2t + \frac{1}{2} i = \frac{1}{2} i + e^{2it} \quad (2)$$

$$\int_{\gamma_2} \frac{dz}{z} = 2\pi i \cdot n(\gamma_2, 0) = 2\pi i \cdot 2 = \underline{\underline{4\pi i}}$$